1. Theoretical Background

Equity models can take several forms, depending on the situation to which the model will be applied. Asset pricing models take the form of either equilibrium or no-arbitrage models. Cash flow (real world) models can take either of these forms as well, but are used to obtain information about the distribution of future returns rather than to price financial instruments.

Equilibrium pricing models make assumptions about the environment driving equity prices and therefore require some calibration to make the model match available market prices. No-arbitrage pricing models, value financial instruments with reference to other assets whose market prices are known so as to be consistent with available market prices.

When considering equity models, it is useful to understand the efficient market hypothesis, as it comes into play in the assumptions underlying some model structures. EMH is attributed to Eugene Fama in the 1960s. It takes three forms:

1. In the "weak" form, all past market prices and data are fully reflected in securities prices.
2. In the "semistrong" form, all publicly available information is fully reflected in securities prices.
3. In the "strong" form, all information is fully reflected in securities prices. In other words, even insider information is of no use.

This hypothesis has generated much discussion as to how efficient markets really are, and to what extent savvy market participants can acquire pertinent information leading them to outperform their peers.

Equity models are generally built under the assumption that equity prices follow a stochastic process, meaning the prices evolve over time in an defined manner, subject to a random innovation. Often models are further constrained by a Markovian assumption, where future stock prices depend only on today's market and the history of the process has no bearing on future equity returns. Assuming equity prices follow a Markov process is consistent with the weak form of the efficient market hypothesis. A Wiener process, also called Brownian motion, is a special type of Markov process. Brownian motion of the underlying asset is one of the structural assumptions of the Black-Scholes equation, the mathematical centerpiece for derivatives pricing.

It is up to the individual practitioner to investigate further to decide on an appropriate model type for the application at hand. Equilibrium and no-arbitrage pricing models each have advantages. The actuarial need for a model, however, may require an appropriate investment return model instead of a pricing model. A review of the following resources provides a good start toward making a sound model choice.

- Market efficiency (or inefficiency) and random walks

  - Bernstein [1996] gives a non-technical introduction to the competing views of behavioral economists and proponents of market efficiency.

  - Campbell et al [1997], §1.5 and Chapter 2, describe the efficient market hypothesis and summarizes some recent empirical tests. They draw a distinction between short and long return-horizons, and conclude that short-horizon (weekly or monthly) returns may be predictable to some degree. However, studies of long-horizon returns are hampered by very small sample sizes and it is difficult to draw firm conclusions from the data.

- Exley [2002] examines biases in models used to predict share prices and currency movements.
- Nefti [2000] provides a thorough treatment of the mathematics leading up to the Black-Scholes partial differential equation.

- Pricing models
  - Equilibrium models
  - No-arbitrage (risk-neutral) models

- Panjer et al [1998], §4.3, explains the distinction between these two approaches. The purpose of no-arbitrage (risk-neutral) models is to value financial instruments consistently with other assets whose market prices are already known. These models do not require assumptions about future expected returns. Equilibrium models, on the other hand, make structural assumptions about economic variables and use these assumptions to derive asset prices. In general, these prices will not automatically agree with current market prices.
- Hull [2000], §11.6 explains the principle of risk-neutral valuation in the context of deriving the Black-Scholes option pricing formula.
- Arnott and Bernstein [2002] estimates the forward-looking equity risk premium, concluding that it is much lower than historical excess returns. This would be relevant in the context of equilibrium models.
- Reilly and Brown [2002], Chapters 8 to 10, gives an introduction to Markowitz portfolio theory and the Capital Asset Pricing Model (an example of an equilibrium model)

- Investment return models, i.e., real-world projection models

- Exley [2002] examines returns on stock market or currency expressed as a risk free return plus a risk premium.
- Panjer et al [1998], §9.1 and 9.4.3 outlines the purpose of these models. In general, the objective is to start with current asset prices and simulate future investment returns. The model may make structural assumptions about the relationships between economic variables and asset returns, or these relationships may be estimated empirically based on historical data.

2. Modeling Returns

The familiar independent lognormal (ILN) model is simple, tractable and easily parameterized under both the risk neutral and real world probability measures. It has clear links to the classic Black-Scholes option-pricing model. A thorough understanding of the lognormal model is needed for many more advanced applications.

More sophisticated equity return models attempt to overcome the primary deficiencies of the lognormal model (constant volatility, symmetry about the mean) by introducing new parameters and/or random innovations. These models generally fit the historic data (real-world) and/or observed prices (risk neutral) much better, but the added complexities typically require more care in implementation and present greater challenges in parameter estimation and 'fitting'. Estimating and modeling correlations among the state variables is a significant theoretical and practical challenge and should not be taken lightly.

Since many problems effectively lead to Monte Carlo simulation for a practical solution, the robustness and reliability of the random number generator is crucial. The generator is so fundamental to the rest of the process that it cannot be the weak link in the chain of techniques that comprise Monte Carlo simulation

State price deflators, or risk-adjusted stochastic discount factors, are convenient constructions that permit fair market valuation using the real-world (*P*-measure) distribution rather than the risk neutral *Q*-measure. If available, the deflators offer the practical advantage of requiring only a single set of cash flow projections under the real-world scenarios – projections that are needed for most actuarial applications.

Of particular mention due to its comprehensiveness, Willmott [2000] is an authoritative 2-volume set that rivals the classic Hull textbook in terms of coverage and applicability. There is a significant amount of theory and mathematics, but enough real-world examples to convey key points and demonstrate applicability to the practitioner. The first volume covers derivatives, path dependency and extensions of Black-Scholes. The second volume covers interest rates & products and risk measurement & management (e.g., portfolio management, asset allocation, VaR, etc.). There is an excellent section on miscellaneous topics (e.g., Orange County, LTCM) and a very thorough treatment of numerical methods. There is a slightly tongue-in-cheek executive summary in the Appendix that provides the needed math background. The bibliography is extensive.

- Risk-neutral (*Q*-measure) models
  - Log-normal model

- Hull [2000], Chapters 10 and 11, describes the lognormal model, which is one of the simplest and most widely used models of stock price returns. He starts with a discrete-time version and then derives a version in continuous time (infinitesimally small time-steps). He describes both the real-world version of the model and the risk-neutral version. . Important concepts include Ito's lemma and the Markov property. A general description of risk neutral valuation is provided, although the reader must turn elsewhere for a more rigorous treatment.

- Campbell et al [1997], §9.1 is another treatment of the same topic.

- Willmott et al [1995] is a good introductory text that describes the modeling of financial derivatives from an applied mathematician's viewpoint. Partial differential equations are presented, using numerical solutions where appropriate. Some mathematics is assumed, but clear explanations are provided for material that is beyond elementary calculus & probability.

- Haug [1998] is a great reference textbook on option pricing formulas. This is truly a definitive source for information for the practitioner. The book provides comprehensive background, explanations, ready-to-use spreadsheets and precise computer code for virtually every option pricing formula in use today. No more searching through obscure papers and academic journals!

- Implied volatility trees
- Jump models
- Stochastic volatility models

- Hull [2000], §17.3 - 17.10
- Rubinstein [1994]
- Campbell et al [1997], §9.3

These authors describe a number of extensions to the traditional lognormal model of stock price returns. These are presented in a risk-neutral context, i.e., the objective is to price financial instruments consistently with other observed market prices, and not necessarily to provide a realistic description of stock price dynamics. However, extension to the real world *P*-measure is readily accomplished by estimating parameters from historic data. However, it is the parameter estimation that is most unwieldy for many of these models, requiring the use of advanced computational and filtering techniques.

- Real-world (*P*-measure) models
  - Log-normal model
  - Autoregressive models (AR, ARCH, GARCH, etc.)

- Panjer et al [1998], §9.2 and 9.3
- Campbell et al [1997], §12.1 and 12.2
- Panneton [1999]
- Hull [2000], Chapter 15
- Hardy [2001]

These authors describe the lognormal model in a real-world context. They also describe autoregressive (AR) models, which is a general class of models that allow for serial correlation in returns. A basic assumption of the lognormal model is that successive returns are independent and identically distributed, but in practice there is strong evidence of serial correlation in volatility, i.e. large returns (of either sign) tend to be followed by more large returns (of either sign). AR models capture this effect.

- Wilkie model
  - Wilkie [1995]
  - Panjer et al [1998], §9.4

  The Wilkie model is an integrated multi-variate model of stock prices, dividend yields, interest rates and inflation designed for long-term projections. The underlying stochastic variables are assumed to be normally distributed with constant volatility. One implication of the model is that equity returns exhibit mean reversion over long investment horizons. The traditional model uses an annual time step, making it unsuitable for certain applications. Attempts to produce intra-year results (e.g., use of a Brownian bridge) add complexity.

- Regime-switching models

  Unlike AR models, under which current period volatility depends on prior period returns, regime-switching models assume that volatility depends on an unobserved state variable that randomly switches between different regimes. For Canadian equity returns this approach seems to provide a better fit to historical data than AR models.

  - Hardy [2001] provides an excellent treatment of the regime-switching lognormal model. The paper is intended for practitioners, but appropriately develops the underlying theory. The practical issues of parameter estimation, model fit and application are thoroughly addressed. Realistic case studies are well structured and thoughtfully presented. An indispensable reference, especially for actuaries dealing with real-world valuation and projection problems.
  - CIA Task Force on Segregated Fund Investment Guarantees [2001] provides a very good overview of equity modeling in the context of valuing the risks of investment guarantees on separate account variable annuity (insurance) products. The paper is decidedly non-technical, but addresses a wide range of issues relevant to the practicing actuary, including significant commentary on the liability/product models.

- Historical simulation
  - Hull [2000] §14.8
  - Hull and White [1998]
  -

Historical simulation bypasses the difficulties of fitting a parametric model to historical data by sampling directly from the data. By definition, this accurately reflects the historical probability distribution as well as relationships between different markets. A disadvantage is that historical data may not be relevant under current market conditions (e.g., there may have been a break in economic regimes). The paper by Hull and White describes a technique that addresses this problem.

- State price deflators

State price deflators are risk-adjusted discount factors that permit fair market pricing using the realistic *P*-measure. This can have practical computational advantages whenever real-world projections are required. Importantly, the deflators depend on the underlying *P*-measure model, not on the cash flows. Unfortunately, it is not always computationally feasible to construct the deflators for some high-dimension problems under certain models.

The deflators should not be interpreted as economic variables, but rather as mathematical constructs that permit fair value calculations. Just as the individual scenarios in a risk neutral simulation have no particular significance, the scenario deflators should not be perceived as having any "physical" meaning.

Jarvis [2001] provides a general introduction to the use of deflators for market consistent valuation.

- Correlations and models of multiple markets

Correlation is one of the most important issues in real world modeling of insurance assets and liabilities.

- Panjer [1998] briefly discusses the topic of generating multi-variate normal random variables (§9.2.7) and highlights the importance of Cholesky decomposition. These methods are relevant to many Markovian models, including stochastic volatility and regime-switching processes. Panjer does not discuss the practical problems of Cholesky methods for multi-dimensional problems, but defers to the wide array of available textbooks on numerical methods.
- Jäckel [2002] highlights the importance and challenges in modeling correlation (more generally, co-movement) and introduces various measures for co-dependence and techniques for 'salvaging' a correlation matrix.
- Press et al [1993] give some practical algorithms for decomposing matrices, including how to deal efficiently with sparse linear systems.

- P-measure vs. Q-measure models

Economic models are used for a variety of purposes, including asset and liability valuation, reserve and capital adequacy testing; product pricing; etc. *P*-measure models are used for cashflow projections, while *Q*-measure models are used for 'market' pricing.

- Panjer [1998] discusses why *Q*-measure models are synthetic constructs and thus are inappropriate for generating future economic scenarios (§9.2.7).
- Hull [2000] Chapter 9 addresses binomial trees and the assumptions inherent in risk neutral valuations.
- CIA Task Force on Segregated Fund Investment Guarantees [2001] discusses the applicability of real world and risk neutral models in various situations.

- Random number generation

The topic of random number generation is fundamental to Monte Carlo simulation. Press et al [1993] give an excellent treatment, as does Jäckel [2002]. The first consideration to address is whether there is any alternative to stochastic simulation. The three basic alternatives are (1) analytic solutions (e.g., Black-Scholes), (2) Lattices and (3) Quasi-Monte Carlo Techniques (e.g., Low Discrepancy Sequences).

In practice, there is often no substitute for large scale Monte Carlo simulation. The usual technique involves generating uniform (0,1) quasi random numbers and then transforming using the inverse of the probability distribution function of the required random variate-type (e.g., Normal). Where correlated random numbers are required, the standard Cholesky decomposition is then applied (subject to constraints). Hence, the decomposition and inverse cumulative distribution function routines are just as important as the U(0,1) generator itself.

Practitioners should be aware of the DIEHARD battery of tests developed by Dr. George Marsaglia (DIEHARD test programs are available by free download).

- Press et al [1993], Chapter 7, warns against reliance on built-in random number generators ("the historical record is nothing if not appalling") and describes several practical alternatives.

- In Chapters 7 – 10, Jäckel [2002] gives a good overview of pseudo-random number generators and low discrepancy sequences. Whatever these sections lack in detail is more than made up in the insightful and highly readable commentary. Particular praise is heaped on the Mersenne Twister, a readily available algorithm with extremely high periodicity. Jäckel emphasizes the importance of the generator in the technique chain for Monte Carlo simulation. Chapters 9 and 10 discusses the important practical topics of non-uniform variates (transformation of the generated U(0,1) values) and variance reduction techniques.

3. Measuring Risk

Companies need to develop ways to measure risk effectively and use various risk measures that will provide a good snapshot of their current risk profile/position. As with any modeling, care needs to be taken in developing assumptions, especially the "dynamic" assumptions that drive the variability of results. These include policyholder behavior, efficiency of option exercise, relationship between bond and equity returns, etc. Sensitivity testing of assumptions is necessary to understand the range of results and to build better business models. The following references outline various risk measures and measurement methods that are in use today. Risk measures that are more specific to insurance products may still need to be developed.

- Methods
  - Stochastic/Monte Carlo Simulations
  - Stress-Testing

- Hull [2000] §14.9
- Jäckel [2002] is entirely devoted to "Monte Carlo Methods in Finance". Common tools and techniques are presented and supplemented with theory and mathematics where appropriate. It is a highly readable book for the novice and does not delve into the highly advanced topics, but suggests alternative references. There is even a 'miscellaneous' chapter on the practical issues related to implementation.

Stress testing is a complement to VaR. It is an estimate of the impact of specified extreme scenarios that might in practice occur from time to time, even though they may be virtually impossible according to the probability distributions underlying VaR calculations.

- Measures
  - Value-at-Risk, Earnings-at-Risk
  - Parametric VaR
  - Alternative risk measures (e.g. CTE, Probable Maximum Loss, Sharpe Ratio, etc.)

- Hull [2000] Chapter 14, introduces Value-at-Risk (VaR). This is an attempt to summarize the total risk in a portfolio of financial assets as a single confidence limit, i.e. "we are *X* percent certain that we will not lose more than *V* dollars in the next *N* days". This chapter outlines some common techniques for estimating VaR.
- Nefti [2000] gives an introduction to extreme value theory and provides empirical evidence that this is more robust approach to VaR than traditional methods.
- Wirch and Hardy [1999]
- CIA Task Force on Segregated Fund Investment Guarantees [2001]

VaR is widely used and easy to understand, but it has some defects, both theoretical and practical. These papers describe alternatives that avoid these problems. One popular alternative is the Conditional Tail Expectation (CTE), which is an estimate of expected losses in excess of the VaR limit (sometimes called expected shortfall or Tail-VaR).

4. Managing Risk

When companies develop methods to measure exposure and understand their risk profile, they also need appropriate tools to assess and manage the risks within acceptable levels. Equity exposure is becoming an increasing portion of the total risk faced by many insurers due to the popularity of variable insurance and annuity products with guarantees. The recent downturn in the stock markets further emphasizes the importance of risk management, especially in the product design and pricing phase.

The following references outline general risk management tools and strategies specifically geared towards insurance products such as equity-linked annuities and variable policies. Care should be taken when utilizing dynamic hedges since it can generate significant losses if not properly understood or executed.

- Passive risk management
- Active risk management (static and dynamic hedging)
- Static risk management for EIA
- Static risk management for EIL
- Static and dynamic risk management and other investment issues for equity-indexed products
- Risk management for variable annuities

- Hull [2000] Chapter 13, describes the "Greeks" (delta, gamma, vega, etc.), each of which measures a different dimension of price sensitivity. The goal is to manage the Greeks with acceptable tolerances.
- *8V-300-00 Equity-Indexed Annuities (EIA) – New Territory on the Efficient Frontier* provides a description of EIA products and associated investment strategies for the equity put & call risks. The note focuses on static hedging using options.
- *8V-301-00 Equity-Indexed Life (EIL) Products* provides a description of EIL products and associated 'static' investment strategies for the equity risks using index options.
- *8V-312-00 AAA Final Report of the Equity-Indexed Products Task Force* offsets a complete description of equity-indexed products and associated investment issues. The requirements and mechanics for various static and dynamic hedging programs are discussed.
- *8V-313-00 Variable Annuities - "No Loss" Proposition* offers a description of the risks on variable annuities and the various risk management tools available, including running the risk, static and dynamic hedging, reinsurance, securitization, making markets and integrated risk management.
- Manistre [2001] provides an excellent overview of how financial economics can be used to value and manage the risks on universal life policies, in contrast to more traditional actuarial techniques.
- Hardy [2003] offers a superb and comprehensive treatment of modeling and risk management for equity-linked insurance and annuities with investment guarantees. An invaluable resource to insurance and financial professionals alike.

5. Other Issues

Not all the issues that occur in modeling and managing equity risk form are dealt with in publicly available references. Following is a discussion of some of these issues.

- Relationship between stock indices and actively-managed mutual funds

Much of the published econometric research on equity returns is based on major stock market indices, e.g. the S&P500. However, insurers' equity risk often comes in the form of exposure to a variety of different actively-managed mutual funds (or separate accounts). One way to deal with this is to start by modeling a limited number of market indices, then model each mutual fund's returns as a linear combination of index returns. A useful starting point is Sharpe [1992]. However, Sharpe is mainly concerned with performance analysis as opposed to risk management, and his technique should be modified if used for the latter purpose. In particular, if his technique is used without modification it may fail to capture the risk that a specific mutual fund's performance may differ significantly from its benchmark index. Notably, Carrière and Hill [2001] use stochastic models for mutual fund returns to explain the risks associated with the net incremental return over a benchmark due to active investment practices.

- Model validation and back-testing

A cornerstone of the model-based approach to capital for banks' trading books is that internal models should be subject to routine back testing. For example, if VaR is modeled at the 99th percentile over a 1-day horizon, then actual one-day profits and losses should not exceed the modeled VaR limit significantly more often than once every 100 trading days.

Life insurers are concerned with much longer risk horizons than banks' trading books, and direct back-testing of modeled confidence limits is not generally feasible (since hundreds or thousands of years of data would be needed). To date there has been little published research on performance testing of long-term equity risk models, but there has been some research on indirect testing of long-term model performance in other contexts – for example, in the context of credit risk models, where long time horizons are also an issue.

Two techniques that are sometimes used are as follows:

- Compare the modeled distribution with actual outcomes at other points across the distribution, not just the tail-risk percentiles. For example, there may not be enough data to compare the modeled 99th percentile with actual outcomes, but there may be enough data for comparisons at lower percentiles.
- Compare the modeled distribution with actual outcomes at shorter time horizons. Back testing of model results may not be feasible for a 10-year risk horizon, but if the same model can generate results for a 1-year risk horizon then it may be feasible to compare the latter with actual outcomes.

Both these techniques are implicit in the "calibration criteria" proposed in the report of the CIA Task Force on Segregated Fund Investment Guarantees [2001], except that task force report was mainly concerned with in-sample model performance, as opposed to out-of-sample back-testing.

- Markov Chain

Markov processes are often assumed in equity modeling because they tend to simplify the calculations and ease the burden of parameter estimation based on empirical data. These processes assume that equity returns only depend on today's state variables, and past prices have no influence on future performance. This idea is reasonably discussed in Hardy [2001], providing basic coverage not significantly enhanced by reviewing other literature on Markovian processes in equity modeling.

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