Duration Defined

Duration is a risk management metric that measures the sensitivity of a financial security's value to changes in interest rates. There are many different and seemingly contradictory definitions of duration in the actuarial and financial literature. Our purpose here is to outline a general framework for understanding the duration concept and show how each of the historical definitions follows as a consequence.

General Definition

For any security or portfolio of securities S with current value P, the **duration $D_{S,r}$ of S with respect to the interest rate r** is

$$D_{S,r} = (-1/P)\ (\partial P/\partial r)$$

where $\partial P/\partial r$ denotes the partial derivative of P with respect to r. In descriptive terms, duration is the negative of the percentage change in price P for an arbitrarily small change in interest rate r, or simply the interest rate elasticity of price.

Although it may not be readily apparent from this definition, the duration depends in general on the time to maturity of the underlying security, the term and compounding frequency of the interest rate, the timing and interest rate sensitivity of future cash flows associated with the security, and possibly the current value of the security itself. Hence, duration is only a meaningful concept when the security and the interest rate are clearly specified. This is illustrated in the following examples.

Example 1: Consider a default-free option-free zero coupon bond with time to maturity t, maturity value M, and current price P. Let r denote the per annum continuously compounded interest rate implied by market conditions. Then

$$P = M\ e^{-rt}$$

and the duration of this security with respect to r is

$$D_{S,r} = (-1/P)\ (\partial P/\partial r) = (-1/(M\ e^{-rt}))(-t\ M\ e^{-rt}) = t.$$

Hence the duration of a default-free option-free zero coupon bond with respect to the implied per annum continuously compounded interest rate is simply the time to maturity.

Example 2: Consider the default-free option-free zero coupon bond of the previous example and let y denote the implied per annum interest rate compounded k times per year. Then

$$P = M\ (1+y/k)^{-kt}$$

and the duration of the bond with respect to y is

$$D_{Syr} = (-1/P)\ (\partial P/\partial y) = -M^{-1}\ (1+y/k)^{kt}\ (M\ (-kt)(1+y/k)^{-kt-1}(1/k)) = t/(1+y/k).$$

This is clearly different from the duration $D_{S,r}$ with respect to the interest rate in continuous form and underscores the fact that duration is only meaningful when the compounding frequency is specified. Note that $D_{S,r}$ and $D_{S,y}$ are connected by the relationship

$$D_{S,y} = D_{S,r} (\partial r/\partial y),$$

which follows from the chain rule of single variable calculus.

Example 3: Consider a portfolio of n default-free option-free zero-coupon bonds with respective maturity values $M_1, \ldots, M_n$ and times to maturity $t_1, \ldots, t_n$. Suppose that the yield curve expressed in continuously compounded form is flat and that only parallel shifts in the continuous yield curve are possible. (These are clearly idealized --- and unrealistic --- assumptions.) Let r denote the per annum continuously compounded interest rate for each term to maturity and let $P_1, \ldots, P_n$ denote the respective prices of the bonds in the portfolio. Then

$$P_i = e^{-rt_i} \quad \text{for } i = 1, \ldots, n$$

and the duration of the portfolio is

$$\begin{aligned} D_{S,r} &= (-1/(P_1 + \ldots + P_n))((\partial/\partial r)(P_1 + \ldots + P_n)) \\ &= (-1/(P_1 + \ldots + P_n))(-t_1 e^{-rt_1} - \ldots - t_n e^{-rt_n}) \\ &= (-1/(P_1 + \ldots + P_n))(-t_1 P_1 - \ldots - t_n P_n) \\ &= (P_1/(P_1 + \ldots + P_n))\, t_1 + \ldots + (P_n/(P_1 + \ldots + P_n))\, t_n. \end{aligned}$$

Hence, the duration of a portfolio of default-free option-free zero-coupon bonds with respect to a continuous interest rate, assuming a flat yield curve and only parallel shifts possible, is a weighted average of the bonds' times to maturity with weights given by the relative values of the bonds in the portfolio.

Historical Note

For any bond with scheduled cash flow payments $P_1, \ldots , P_n$ at times $t_1, \ldots, t_n$ and effective yield y, the **Macaulay duration** is

$$D^{Mac} = \left(\sum_{I=1}^{n} P_i (1+y)^{-t_i} t_i \right) / \left(\sum_{I=1}^{n} P_i (1+y)^{-t_i} \right)$$

and the **modified duration** is

$$D^{mod} = D^{Mac} / (1+y).$$

For a bond with k interest payments per year and yield y compounded k times per annum, the modified duration is

$$D^{mod,\, k} = D^{Mac}/(1+y/k).$$

Note that the Macaulay duration is simply the weighted average of the payment times $t_1, \ldots, t_n$ with weights given by the relative sizes of the present values of the individual cash flows.

From Examples 1, 2, and 3, it should be clear that the Macaulay duration is simply the duration (as defined under the heading "General Definition" earlier in this note) with respect to a continuous interest rate under the assumptions that

1. the bond is default-free and option-free,
2. the yield curve in continuous form is flat,
3. only parallel shifts are possible.

Similarly, the modified duration is simply the duration with respect to an interest rate with a compounding frequency of k per annum and satisfying these three assumptions.

The Scottish actuary Frederick Macaulay introduced the concept of duration in 1938, primarily to measure the "average time to repayment" for a coupon-paying bond. This explains why the metric has the name "duration". It was only later in the work of Redington on interest rate immunization that the duration was recognized more fully as a measure of interest rate sensitivity.

Duration, as generally defined in this note, is also applicable to bonds with embedded options (such as callable bonds) and to situations where the yield curve is not flat and does not necessarily move in parallel shifts. This is illustrated in the following two examples.

Example 4: Consider a portfolio of two default-free, option-free zero coupon bonds with respective maturity values M_1, M_2 and times to maturity t_1, t_2. Suppose that the yields, in per annum continuously compounded form, for default-free option-free securities with maturities t_1, t_2 are r_1, r_2 respectively and let P_1, P_2 denote the respective prices of the bonds in the portfolio. Then

$$P_1 = e^{-r_1 t_1},$$
$$P_2 = e^{-r_2 t_2},$$

and the duration of the portfolio with respect to r_1 is

$$D_{S,r1} = (-1/(P_1+P_2))\,(\partial/\partial r_1)(P_1+P_2)$$
$$= (-1/(P_1+P_2))(-t_1\, e^{-r_1 t_1} - t_2\, e^{-r_2 t_2}\, (\partial r_2/\partial r_1))$$
$$= (P_1/(P_1+P_2))\, t_1 + (\partial r_2/\partial r_1)\, (P_2/(P_1+P_2))\, t_2.$$

Similarly, the duration of the portfolio with respect to r_2 is

$$D_{S,r2} = (\partial r_1/\partial r_2)\, (P_1/(P_1+P_2))\, t_1 + (P_2/(P_1+P_2))\, t_2.$$

Note that if $\partial r_1/\partial r_2 = 1$ and $\partial r_2/\partial r_1 = 1$ then the durations with respect to r_1 and r_2 are equal and reduce to the duration value calculated in Example 3, where it was assumed that yield curve shifts are parallel. At the opposite extreme, if $\partial r_1/\partial r_2 = 0$ and $\partial r_2/\partial r_1 = 0$ then the durations with respect to r_1 and r_2 reduce to

$$D_{S,r1}^{*} = (P_1/(P_1+P_2))\, t_1,$$
$$D_{S,r2}^{*} = (P_2/(P_1+P_2))\, t_2.$$

Durations of this type are known as *partial* durations because they measure interest rate sensitivity under the assumption that only one rate in the term structure can change.

<u>Definition of Partial Duration</u>

Consider a security S with current value P and cash flows at times $t_1, \ldots, t_n$. Let $r_1, \ldots, r_n$ be the interest rates on the yield curve for this security with terms to maturity $t_1, \ldots, t_n$. The **partial duration $D_{S,\, ri,\, ti}$ of S with respect to interest rate r_i and term to maturity t_i** is

$$D_{S,\, ri,\, ti} = (-1/P)(\partial P/\partial r_i),$$

where the partial derivative is calculated under the assumption that $r_1, \ldots, r_n$ are independent, i.e., $\partial r_j/\partial r_i = 0$ for $j \neq i$.

Partial duration is a useful metric when one wishes to measure the sensitivity of a portfolio to movements in various parts of the yield curve. By matching partial durations of asset and liability portfolios it is possible to obtain a degree of protection against nonparallel shifts in the yield curve (see the Applications section at the end of this note). However, one should keep in mind that the rates corresponding to various terms to maturity are not in fact independent, and so additional tools are required to fully quantify the sensitivity of a portfolio to nonparallel shifts.

Example 5: Consider a default-free zero-coupon bond with maturity value M and time to maturity t, which is callable at price C during the first half of the bond's life (i.e., the issuer has the option to buy back the bond by paying the bondholder the amount C at any time during the first half of the life of the bond) and let P be the current value under a particular term structure. Determining the duration for such a security is more challenging than in the previous examples since there is generally no explicit formula for the value of the security (i.e., the value of P) when the future cash flows are not fixed but depend on the evolution of the yield curve over time. To determine the value of P in such cases, one needs to use a stochastic interest rate model and some basic option pricing theory. Not surprisingly, the value of P obtained depends on the interest rate model used, even when the currently observable term structure is the starting point for all models. Leaving aside this issue for the moment, the duration can be approximated by the formula

$$D \approx (-1/P)\ (\ (P^+ - P^-)/(r^+ - r^-)\ )$$

where P^+ is the value of the security when the pricing model is run with an initial short term rate of r^+ in the term structure (where r^+ slightly greater than the currently observable value of the short rate) and P^- is the value of the security when the pricing model is run with an initial short rate of r^- (where r^- is slightly less than the currently observable short rate). The duration can also be estimated with respect to a rate other than the short rate. However, one should keep in mind that the interest rate model used will determine the extent to which one can arbitrarily change the initial value of a rate other than the short rate (and hence the extent to which one can estimate the duration with respect to a rate other than the short rate). Moreover, one must keep in mind that a change to the initial value of any particular interest rate will generally result in a change to the entire yield curve, not just a single rate.

From this example, it is apparent that the duration for a security with interest sensitive cash flows (due for example to embedded options) depends on both the interest rate model used to price the security and the way in which a perturbation of the observable yield curve is selected. Durations estimated in this way are known as *effective* durations.

Definition of Effective Duration

Consider a security whose future cash flows are sensitive to interest rate movements. For such a security, the **effective duration** is the quantity

$$D \approx (-1/P)\ (\ (P^+ - P^-)/(r^+ - r^-)\ )$$

where P is the value of the security using a predetermined stochastic interest rate model with initial yield curve taken to be the currently observable yield curve, P^+ is the value of the security using the same interest rate model with initial yield curve perturbed in the positive direction, and P^- is the value of the security using the same interest rate model with initial yield curve perturbed in the negative direction.

Statistical Duration

There is another duration measure that is particularly useful when analyzing the interest rate sensitivity of securities such as equities whose cash flows are not well defined. This type of duration is known as ***statistical duration*** and is calculated by regressing the historical percentage changes in the security's value on the corresponding historical interest rate changes:

$$(DP/P) = a - D\,(Dr) + e$$

where D is determined by minimizing a sum of squared error terms in the standard way. Note that the statistical duration depends on the historical time period selected, the time step size, and the interest rate used as a dependent variable (e.g., short-rate, one-year rate, etc). In cases where duration can be calculated directly from a formula or using interest rate and option pricing models, the statistical duration provides a valuable check on one's results. On the other hand, in situations where such formulas do not exist (e.g., duration of equities), the statistical duration may be the only meaningful measure of interest rate sensitivity.

Applications of Duration

Weaknesses of Duration as a Risk Measure

References